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Gravitational instability of a viscous fluid in a magnetic field

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The instability of a viscous fluid between two infinite vertical plates and heated from below in the presence of a magnetic field perpendicular to the plates is investigated, and the most critical stability boundary in the space of the Rayleigh number R, Hartmann number M, and the horizontal wave number a is determined. It is found that the most unstable mode is a symmetric mode with zero wave-number, and that for any M the fluid is unstable for any non-zero R, however small.

1. Introduction

The problem studied concerns the stability of a viscous fluid contained between two vertical infinite plates and heated from below in the presence of a uniform magnetic field perpendicular to the plates. The mean temperature field is given by

$$\overline{T} = T_0 + \beta z, \tag{1}$$

with z measured vertically upward, and β indicating the (negative) vertical gradient of the temperature. The mean density is then

$$\overline{\rho} = \rho_0(1-\alpha\beta z), \tag{2}$$

in which α is the thermal expansivity. The gradient of the hydrostatic pressure $\overline{p}$ is then

$$d\overline{p}/dz = -g\rho_0(1-\alpha\beta z), \tag{3}$$

in which g is the gravitational acceleration. The quantities T_0 and ρ_0 are the values of $\overline{T}$ and $\overline{\rho}$ at $z = 0$. The only component of the magnetic field when the fluid is undisturbed is $\overline{H}_x$, which is in the direction normal to the plates. Its magnitude is denoted by H_0. The direction of y is therefore horizontal and parallel to the plates.

The problem with zero magnetic field has been considered by Ostrach (1955), Yih (1959), and Wooding (1960). Ostrach considered purely vertical motion. Yih considered the stability of disturbances with wavelengths in the z-direction, proved the validity of the principle of exchange of stabilities, and showed that the most unstable mode is the one with infinite wavelength, or zero wave-number in the direction of the vertical. The results of Ostrach and Yih, for both symmetric and antisymmetric convection, are actually in agreement, although Yih did not realize that the number 31·29 for antisymmetric convection had already been given in Ostrach's paper. However, Wooding (1960) showed that both Ostrach

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and Yih had missed the most unstable mode. By considering motion in the y-direction he showed that, for motion symmetric with respect to x, the critical Rayleigh number (to be defined later in this paper) is indeed zero, corresponding to zero wave-number in the y-direction. Zero indeed is a root of the secular equation obtained by Ostrach and Yih for symmetric convection. But its significance escaped them both!

Wooding gave, for the case of zero magnetic field, an expansion of the Rayleigh number in powers of the wave-number in the y-direction, here indicated by a to distinguish it from the customary symbol α for the thermal expansivity,

$$R = 3a^2\{1 + \tfrac{8}{21}a^2 + O(a^4)\}. \tag{4}$$

Wooding's demonstration that the fluid is unstable for all Rayleigh numbers is beyond doubt, so long as the plates are infinite in the y-direction.

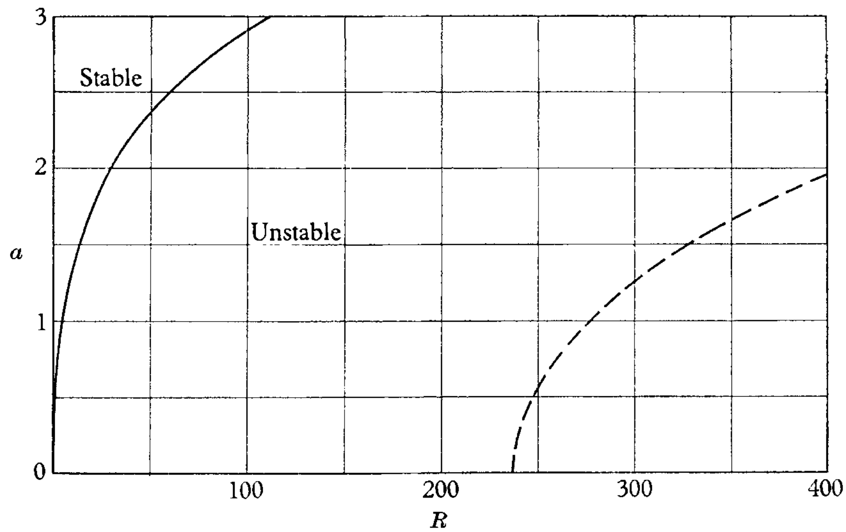

FIGURE 1. The solid line is the true stability boundary for $M = 0$. The broken line is Dunwoody's stability boundary.

The problem stated in this section has been studied by Dunwoody (1964). His results for symmetric convection are given in broken lines in figure 1 for the case of zero magnetic field and in figure 2 for the case of a non-zero magnetic field (with M indicating the Hartmann number and $a = 0$ in this case). It is clear that Dunwoody's curves do not pass through the origin and that his results are at variance with Wooding's finding. Dunwoody regards Wooding's result for $a = 0$ and $H_0 = 0$ (or $M = 0$) as correct. But he seems to consider that result to be an isolated incident, and has ignored the expansion (4) entirely. The 'stable' regions in his figures 2 and 4 are not stable regions if Wooding's results are correct. If, as Dunwoody agrees, the fluid is incipiently unstable at $R = 0$ and $M = 0$, how can it be stable for $M = 0$ and $0 < R < 237{\cdot}6$, as indicated by his figures 2 and 4? To put it another way, how can R jump from zero (given in Dunwoody's table 1) to $237{\cdot}6+O(\epsilon^2)$ as a changes from zero to ϵ? Or from zero (given in Dunwoody's table 2) to $237{\cdot}6+O(\epsilon^2)$ as M changes from zero to ϵ? These jumps are indicated in his figures 2 and 4, and implied in his tables 1 and 2.

These puzzling points led the writer to investigate the present problem anew. It will be shown that Dunwoody missed the most unstable mode of symmetric convection, that the results for that mode are consistent with Wooding's finding for $M = 0$, and that, contrary to Dunwood's conclusion, the most critical symmetric mode is more unstable than the most critical antisymmetric mode.

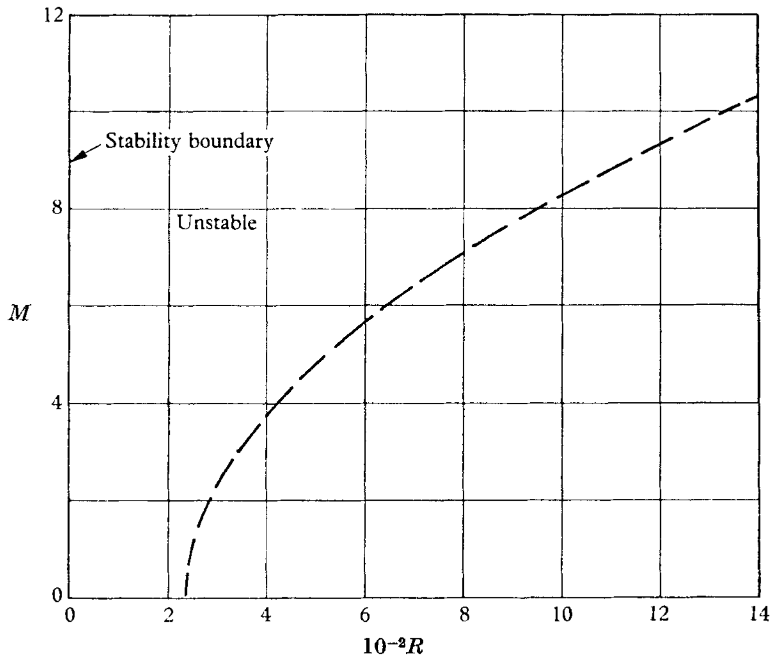

FIGURE 2. The M-axis is the stability boundary for $a = 0$. The broken line is Dunwoody's stability boundary.

2. The governing differential system

Since some of the equations in Dunwoody's formulation of the mathematical problem will be referred to in this discussion, the key equations in his paper will be reproduced here. There will be a slight change in notation. The wave-number will be denoted by a instead of α, the thermal expansivity by α instead of ϕ, and the bars over a symbol denote mean quantities here rather than dimensionless variables. In the following, $2d$ denotes the spacing of the plates, ν the kinematic viscosity, κ the thermal diffusivity, t the time, T' the temperature perturbation, w the vertical velocity, p' the pressure perturbation, and H_z the vertical component of the magnetic field. Dunwoody assumes that the velocity components u and v in the directions of x and y are zero, that the only component of the induced magnetic field is H_z, so that

$$H_x = H_0, \quad H_y = 0. \tag{5}$$

Furthermore, the perturbation quantities w, T', and H_z are supposed independent of z, and the co-ordinates

$$x' = \frac{x}{d}, \quad y' = \frac{y}{d}, \quad t' = \frac{\nu t}{d^2},$$

will be used *without* the accents. The exponential time factor will be assumed for the perturbation quantities, so that

$$\frac{wd}{\kappa} = W(x,y)\,e^{\lambda t}, \quad \frac{T'}{\beta d} = \theta(x,y)\,e^{\lambda t}, \quad \frac{H_z}{H_0} = H(x,y)\,e^{\lambda t}, \quad \frac{\partial p'}{\partial z} = \Pi\,e^{\lambda t}. \tag{6}$$

However, Π must be zero, for otherwise p' will be infinite at $z = \pm\infty$, violating the basis of the linear theory. (This rules out the Hartmann flow, which would otherwise be a perfectly acceptable solution of (7), (8), and (9) below, for $\lambda = 0$ and $R = 0$, as a possible neutral mode.)

The linearized equations of motion in the z-direction, of thermal diffusion, and, for the magnetic field are then

$$\lambda W = -R\theta + \nabla^2 W + \eta M^2 \frac{\partial H}{\partial x}, \tag{7}$$

$$P\lambda\theta = \nabla^2\theta - W, \tag{8}$$

$$P\lambda H = \eta\nabla^2 H + \partial W/\partial x, \tag{9}$$

in which $\eta = (4\pi\mu\sigma\kappa)^{-1}$ is the ratio of the magnetic diffusivity to thermal diffusivity, $P = \nu/\kappa$ is the Prandtl number, and $M^2 = \sigma u^2 H_0^2 d^2/\rho_0\nu$ and $R = -\alpha\beta g d^4/\nu\kappa$, are the Hartmann number squared and the Rayleigh number. In the expressions for η and M, μ is the magnetic permeability, and σ the electrical conductivity. The symbol ∇^2 denotes the Laplacian in x and y.

The boundary conditions are: (*a*) the non-slip condition, (*b*) the condition of insulation at the wall, and (*c*) the continuity of H_z at the wall, in which it is assumed to be zero. Thus they are

$$W = 0, \quad \partial\theta/\partial x = 0, \quad \text{and} \quad H = 0 \quad \text{at} \quad x = \pm 1, \tag{10}$$

if the origin of the co-ordinates is located midway between the plates.

The quantity λ is in general complex. Since the principle of exchange of stabilities has been shown to be valid by Dunwoody, for neutral stability it can be taken to be zero. If, further,

$$W(x,y) = f(x)\cos ay, \tag{11}$$

the equations (7), (8) and (9) can be reduced to the single equation in $f(x)$, which is

$$\{(D^2 - a^2)^2 - M^2 D^2\} f = Rf, \tag{12}$$

with D denoting d/dx. The boundary conditions can be reduced to

$$f = 0 \quad \text{and} \quad (D^2 - a^2 - M^2)\,Df = 0 \quad \text{at} \quad x = \pm 1. \tag{13}$$

3. Solution for symmetric convection

The system consisting of (12) and (13) admits even and odd solutions separately. There is no error of omission in Dunwoody's work for odd f, corresponding to antisymmetric convection. We shall concentrate on even f on symmetric convection. The secular equation for that has been correctly obtained by Dunwoody, and is

$$(\epsilon_1^2 - a^2 - M^2)\,\epsilon_1 \tanh \epsilon_1 = (\epsilon_2^2 - a^2 - M^2)\,\epsilon_2 \tanh \epsilon_2, \tag{14}$$

in which

$$2\epsilon_1^2 = 2a^2 + M^2 + B, \quad 2\epsilon_2^2 = 2a^2 + M^2 - B, \quad B = (M^4 + 4M^2a^2 + 4R)^{\frac{1}{2}}. \tag{15}$$

The solutions of (14) for $M = 0$ or for $a = 0$ obtained by Dunwoody are indicated in figures 1 and 2 by broken lines. They are without errors in so far as they satisfy (14). But the most unstable mode has escaped notice, and, as mentioned in the introduction, many puzzling points raised by these broken lines demand explanation. Inspection of (15) shows that $B = 0$ satisfies (14). But this will not do, because the indicial equation of (12) then has a double root, and (14) must be modified because of the solutions in the form of x times a hyperbolic sine. When the modification is made, it is found that $B = 0$ does not satisfy the secular equation.

But, for any M whatever, the equations

$$a = 0 \quad \text{and} \quad R = 0 \tag{16}$$

do satisfy (14). This indicates that R is small if a is non-zero but small. In fact, as will be verified *a posteriori*, $R = O(a^2)$ for small a. Considering small a and *any* M small enough for a four-term expansion of $\tanh \epsilon_1$ or of $\tanh \epsilon_2$ to be sufficiently accurate, we can calculate R from (14). We do not make any stronger assumption concerning M. It may be greater or smaller than a, and M^4 may be greater than R. It turns out that we never have to expand the radical equal to B in (15), because only B^2 or its powers are involved after the factor B has been cancelled. The result in fact is

$$-30a^2 + 10a^4 - 4a^6 + \tfrac{34}{21}a^8 + (M^4 - B^2)\left[-\tfrac{5}{2} + 3a^2 - \tfrac{17}{7}a^4 + M^2 - \tfrac{34}{21}M^2a^2 - \tfrac{17}{168}(3M^4 + B^2)\right] = 0,$$

which involve R and R^2, but no radicals. The final evaluation of R gives

$$R = Aa^2 + Ba^4 + O(a^6), \tag{17}$$

in which
$$A = 315(105 - 42M^2 + 17M^4)^{-1} - M^2,$$

$$B = (10 - 4M^2 + \tfrac{34}{21}M^4)^{-1}\{-10 + 12M^2 - \tfrac{170}{21}M^4 + A(12 - \tfrac{68}{7}M^2 - \tfrac{34}{21}A)\}.$$

For $M = 0$, (17) reduces to (4). For small M, both A and B increases with M. Although (17) has been obtained by taking only four terms in $\tanh \epsilon_1$ and $\tanh \epsilon_2$, and is therefore valid only for small values of M and a, the increase of A and B with M can be expected to hold for any M, however large, on physical grounds. Thus the intersection of the stability boundary with the plane M = constant C

is a curve embracing the R-axis, and doing so more and more closely as C increases.

We have not evaluated R for very large values of M because there has been no need to do so. For large M series expansions of $\tanh \epsilon_1$ and $\tanh \epsilon_2$ are of course impractical. A referee of this paper has shown by assuming $M^4 \gg R$ that $R = Ma^2$ for large M. This does not contradict what has been obtained and said above in the least. In fact it confirms the statement that the stability boundary intersects the plane $M = C$ in a curve that embraces the R-axis more and more closely as C increases.

For $M = 0$, (14) reduces to (3.14) in Dunwoody (1964), or

$$\xi \tanh \xi = \zeta \tan \zeta, \tag{18}$$

with

$$\xi = (R^{\frac{1}{2}} + a^2)^{\frac{1}{2}}, \quad \zeta = (R^{\frac{1}{2}} - a^2)^{\frac{1}{2}}.$$

The (R, a) curve for $M = 0$ is given in figure 1 in solid line. This has been obtained directly from (18) by numerical calculation, by insisting on $\zeta < \frac{1}{2}\pi$. It is evident that it is not only generally more 'critical' than the broken line, but gives a critical Rayleigh number far less than 237·6. For $a = 0$, the neutral-stability curve is simply $R = 0$, or the M-axis, as shown in figure 2. It is again evident that the broken line lies entirely within the unstable region.

The solid line in figure 1 is based on figures kindly provided by Mr S. P. Lin and given in the following table:

a	0·5	1	2	3
R	0·82	4·1	29·5	113·6

To visualize the true stability boundary in the (R, a, M)-space, one may consider it as an infinite sail. The M-axis is the mast and the R-axis the centre-line of the sail boat or the projection of the keel on the deck. The solid line in figure 1 is a curved boom. The positive direction of the R-axis points toward the stern. The broken line in figure 1 is a second curved boom attached to a curved secondary mast indicated by the broken line in figure 2. The 'stern jib' that extends from the second boom to the secondary mast is the stability boundary for one of the infinitely many higher and more stable modes.

Before concluding, it is desirable to clarify the rather puzzling situation that the fluid can be unstable at zero Rayleigh number, even in the presence of a magnetic field. Inspection of (7)–(10), with $\lambda = 0$, reveals that for $a = 0$ the solution is

$$W = 0, \quad H = 0, \quad \theta = \text{const.} \tag{19}$$

provided $R = 0$. For this mode there is no motion and no induced magnetic field, and the definition of θ in (6) shows that T' is equal to a constant (since $\lambda = 0$) times βd. Now for a given fluid R can be zero if β or d is zero. In either case T' is zero. Hence the case of neutral stability corresponding to $a = 0$ and $R = 0$ is characterized by the absence of motion, of induced magnetic field, and of induced temperature. This removes the apparent difficulty of accepting *neutral* stability at zero Rayleigh number, but also raises the question of whether a formal solution characterized by no perturbation of physical quantities at all is qualified to represent a situation of neutral stability. But then we have just shown that in the neighbourhood of such a seemingly insignificant solution are solutions indicating instability at any non-zero Rayleigh number, however small.

We shall make the result $R \to 0$ as $a \to 0$ more palatable by showing that λ is proportional to R. This is done in the following way. As has been said in the last paragraph, for $a = 0$ equations (19) hold. Let the constant for θ be θ_0. As R is increased slightly, Π still must be zero for the reason stated before. But W and H will be different from zero, and θ will be equal to $\theta_0 + \theta_1$. Equations (7)–(9) become

$$\lambda W = -R\theta_0 + W'' + \eta M^2 H',$$
$$P\lambda\theta_0 = \theta_1'' - W,$$
$$P\lambda H = \eta H'' + W'.$$

Now since $\lambda = 0$ at $R = 0$, we expect λ to be small compared with 1 for small R. Thus these equations can be further simplified to

$$0 = -R\theta_0 + W'' + \eta M^2 H', \tag{20}$$
$$P\lambda\theta_0 = \theta_1'' - W, \tag{21}$$
$$0 = \eta H'' + W'. \tag{22}$$

From these it is immediately clear that W and H are proportional to R. In fact, on setting

$$W = RW_1 \quad \text{and} \quad H = RH_1, \tag{23}$$

we can write (20) and (22) as

$$0 = -\theta_0 + W_1'' + \eta M^2 H_1', \tag{24}$$
$$0 = \eta H_1'' + W_1', \tag{25}$$

combination of which yields

$$\theta_0 = \eta H_1''' + \eta M^2 H_1'. \tag{26}$$

Odd as it may appear at first sight, this can be solved with (25) to satisfy the four boundary conditions on W and H in (10). The solution is simply†

$$\left.\begin{aligned} H_1 &= Ax + B\sin Mx, \\ W_1 &= -\eta H_1' + D, \end{aligned}\right\} \tag{27}$$

in which

$$A = \theta_0(\eta M^2)^{-1}, \quad B = -A(\sin M)^{-1} \quad \text{and} \quad D = \eta A(1 - M\cot M).$$

This in fact confirms the adequacy of (23). Now multiplication of (21) by θ_0 and integration between $x = -1$ and $x = 1$ yield

$$2P\lambda\theta_0^2 = -\theta_0 \int W\,dx = -R\int \theta_0 W_1\,dx, \tag{28}$$

since θ_1' vanishes at both limits. Multiplication of (24) by W_1 and integration between the same limits yield

$$\int \theta_0 W_1\,dx = \int W_1'' W_1\,dx + \eta M^2 \int H_1' W_1\,dx,$$

or, upon integration by parts and utilization of (10) and (25),

$$\begin{aligned} \int \theta_0 W_1\,dx &= -\int W_1'^2\,dx - \eta M^2 \int H_1 W_1'\,dx \\ &= -\int W_1'^2\,dx - \eta^2 M^2 \int H_1'^2\,dx. \end{aligned} \tag{29}$$

† This demonstration is based on $M \neq 0$. That for $M = 0$ is similar.

Substitution of (29) in (28) produces, finally,

$$2P\lambda\theta_0^2 = R\int[(W_1')^2+(\eta MH_1')^2]\,dx. \tag{30}$$

Hence λ is proportional to R and is positive for any positive R, however small.

4. Conclusions

We are then in a position to conclude that, whether a magnetic field H_x is present or not:

1. Symmetric convection is more unstable than antisymmetric convection;
2. The fluid is unstable for any non-zero Rayleigh number, however small;
3. For any Hartmann number M, the most unstable mode corresponds to $a = 0$;
4. The (R, a) curve for neutral stability in a plane with constant M embraces the R-axis more and more closely as the value of M increases, giving a smaller and smaller region of instability but keeping zero as the critical Rayleigh number.

Conclusion 3 is in agreement with Dunwoody's conclusion. The other conclusions are new, and are at variance with the results of Dunwoody. All the conclusions are consistent with the results of Wooding (1960) for $M = 0$.

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REFERENCES

Dunwoody, N. T. 1964 *J. Fluid Mech.* **20**, 103.

Ostrach, S. 1955 *50 Jahre Grenzschichtforschung*. Brunswick: Verlag Friedr. Vieweg und Sohn.

Wooding, R. A. 1960 *J. Fluid Mech.* **7**, 501.

Yih, C.-S. 1959 *Quart. Appl. Math.* **17**, 25.